Norwegian University of Science and Technology
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TMA4265 Stochastic processes

Monday 9 August 2010 9:00–13:00

Solutions

(Corrected 24 August 2010)

Problem 1

a) $P(X_2 = 2 \mid X_0 = 1) = \sum_{i=1}^{3} P(X_2 = 2 \mid X_1 = i) P(X_1 = i \mid X_0 = 1) = \frac{1}{3} \cdot \frac{2}{3} + \frac{1}{4} \cdot \frac{1}{3} + \frac{1}{3} \cdot 0 = \frac{11}{36} \approx 0.31$. This is also the $(1, 2)$ entry of the square of the transition matrix.

$P(X_2 \neq 3, X_1 \neq 3 \mid X_0 = 2) = \sum_{i=1}^{2} P(X_2 \neq 3 \mid X_1 = i) P(X_1 = i \mid X_0 = 2) = (\frac{2}{3} + \frac{1}{3}) \cdot \frac{5}{12} + (\frac{5}{12} + \frac{1}{4}) \cdot \frac{1}{4} = \frac{7}{12} \approx 0.58$. This can also be obtained by defining state 3 as absorbing (that is, replacing the last row of the transition matrix by $(0\ 0\ 1)$) and adding the $(2, 1)$ and $(2, 2)$ entries of the square of the transition matrix.

b) Let $u_i = E(T \mid X_0 = i)$, $i = 1, 2, 3$. Then $u_1 = 1 + \frac{2}{3}u_1 + \frac{1}{3}u_2$ and $u_2 = 1 + \frac{5}{12}u_1 + \frac{1}{4}u_2$, giving $u_1 = 39/4 = 9.75$ and $u_2 = 27/4 = 6.75$ (first step analysis). In addition $u_3 = 0$.

c) The chain is irreducible since all states commmunicate. Hence, since the sample state is finite, the chain is positive recurrent. It is possible to return to any state in any number of steps, so the chain is aperiodic. Thus, the limiting distribution exists. Let $\pi_i = \lim_{n\to\infty} P(X_n = i)$, $i = 1, 2, 3$. Then $\pi_1 = \frac{2}{3}\pi_1 + \frac{5}{12}\pi_2 + \frac{1}{3}\pi_3$, $\pi_2 = \frac{1}{3}\pi_1 + \frac{1}{4}\pi_2 + \frac{1}{3}\pi_3$, $\pi_3 = \frac{1}{3}\pi_2 + \frac{1}{3}\pi_3$, and $\pi_1 + \pi_2 + \pi_3 = 1$, giving $\pi_1 = 7/13 \approx 0.54$, $\pi_2 = 4/13 \approx 0.31$, and $\pi_3 = 2/13 \approx 0.15$.

d) $P(X_{n+1} = X_n) = \sum_{i=1}^{3} P(X_{n+1} = i \mid X_n = i) P(X_n = i) \to \frac{2}{3}\pi_1 + \frac{1}{4}\pi_2 + \frac{1}{3}\pi_3 = \frac{2}{3} \cdot \frac{7}{13} + \frac{1}{4} \cdot \frac{4}{13} + \frac{1}{3} \cdot \frac{2}{13} = 19/39 \approx 0.49$ as $n \to \infty$.

Problem 2

If $X_1 = j$, the population will eventually die out if and only if each of the j families started by the members of the first generation eventually dies out. Since the families act independently, and the probability that any paricular family dies out is π_0, $P(\text{population dies out} \mid X_1 = j) = \pi_0^j$. By the law of total probability,

$$\begin{aligned}\pi_0 &= P(\text{population dies out})\\ &= \sum_{j=0}^{\infty} P(\text{population dies out} \mid X_1 = j)P(Z = j)\\ &= \sum_{j=0}^{\infty} \pi_0^j P(Z = j),\end{aligned}$$

which in the present case yields $\pi_0 = \sum_{j=0}^{\infty} \pi_0^j (1-p)^j p = p \sum_{j=0}^{\infty}((\pi_0(1-p))^j = p/(1-\pi_0(1-p))$, so that $(1-p)\pi_0^2 - \pi_0 + p = 0$, giving $\pi_0 = 1$ or $\pi_0 = p/(1-p)$.

When $EZ \le 1$, that is, $p \ge 1/2$, $\pi_0 = 1$. Otherwise, $\pi_0 = p/(1-p)$ (the smallest of the two above solutions).

Problem 3

a) The probability that the first customer is a woman is $\lambda_\mathrm{k}/(\lambda_\mathrm{k} + \lambda_\mathrm{m}) = 0.25/(0.25 + 0.2) = 5/9 \approx 0.56$. The number of women among the first 4 customers has the binomial distribution with parameters $(4, 5/9)$, so the probability that this number is at least 2 is $1 - (\frac{4}{9})^4 - 4 \cdot \frac{5}{9} \cdot (\frac{4}{9})^3 \approx 0.77$.

b) $N(t)$ is Poisson distributed with mean $(\lambda_\mathrm{k} + \lambda_\mathrm{m})t = 0.45t$. $P(N(8) \ge 4) = 1 - P(N(8) \le 3) = 1 - e^{-0.45 \cdot 8} \sum_{i=0}^{3} (0.45 \cdot 8)^i / i! \approx 0.48$. $P(N(8) \ge 4 \mid N(4) = 2) = P(N(4) \ge 2) = 1 - P(N(4) \le 1) = 1 - e^{-4 \cdot 0.45}(1 + 4 \cdot 0.45) \approx 0.54$.

c) We can think of T as the time of the first event of a Poisson process having rate μ. $P(Y = y)$ is then the probability that exactly y events of the customer arrival process (having rate λ) occur before the first event of the independent process having rate μ. Each event that occurs is going to be an event of the customer process with probability $\lambda/(\lambda+\mu)$ or an event of the rate μ process with probability $\mu/(\lambda+\mu)$, so that $P(Y = y) = (\frac{\lambda}{\lambda+\mu})^y \frac{\mu}{\lambda+\mu}$. In particular, $P(Y = 0) = \mu/(\lambda + \mu)$ and $P(Y = 1) = \lambda\mu/(\lambda + \mu)^2$.

Applying the law of total probability to Y conditioning on T will work, too.

d) $X_{\mathrm{k}}(t)$ is a birth and death process because, if $X_{\mathrm{k}}(t) = i$, then the next transition will be to $i+1$ or $i-1$, and the time until the next arrival is exponentially distributed with rate $\lambda_i = \lambda_{\mathrm{k}}$ and is independent of the time until the next departure which is exponentially distributed with rate $\mu_i = i\mu_{\mathrm{k}},\ \ i = 0,\ 1,\ 2, \ldots$.

Let T_i denote the time, starting from state i, it takes for the process to enter state $i+1$. Then $ET_i = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} ET_{i-1},\ \ i \geq 1$. Since $ET_0 = 1/\lambda_0$, expected time to go from state 0 to 2 is $ET_0 + ET_1 = \frac{1}{\lambda_0} + \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1}\frac{1}{\lambda_0} = \frac{2}{\lambda_{\mathrm{k}}} + \frac{\mu_{\mathrm{k}}}{\lambda_{\mathrm{k}}^2} = 14.4$ min.

e) We introduce a continous-time Markov chain having states 00, 01, 10, 02, 11, and 20, the first number denoting the number of women in the shop and the second denoting the number of men. Transition rates out of each state are given by $v_{00} = \lambda,\ \ v_{01} = \lambda + \mu_{\mathrm{m}}$, $v_{10} = \lambda + \mu_{\mathrm{k}},\ \ v_{02} = 2\mu_{\mathrm{m}},\ \ v_{11} = \mu_{\mathrm{k}} + \mu_{\mathrm{m}}$, and $v_{20} = 2\mu_{\mathrm{k}}$.

Let T, T_{01}, and T_{10} denote the time it takes, starting with 0 customers, 0 women and 1 man, and 1 woman and 0 men, respectively, until there are 2 customers in the shop. The expected time to leave state 01 is $1/(\lambda + \mu_{\mathrm{m}})$, and the first transition will be to 00 with probability $\mu_{\mathrm{m}}/(\lambda + \mu_{\mathrm{m}})$, in which case an expected additional time ET is needed to reach 2 customers – otherwise 02 or 11 is reached. Thus,

$$ET_{01} = \frac{1}{\lambda + \mu_{\mathrm{m}}} + \frac{\mu_{\mathrm{m}}}{\lambda + \mu_{\mathrm{m}}} ET = \frac{1}{\lambda + \mu_{\mathrm{m}}}(1 + \mu_{\mathrm{m}} ET).$$

Similarly,

$$ET_{10} = \frac{1}{\lambda + \mu_{\mathrm{k}}}(1 + \mu_{\mathrm{k}} ET).$$

The expected time to leave state 00 is $1/\lambda$, and the transition will be to 01 or to 10 with probabilities λ_{m} and λ_{k}, respectively, so that

$$ET = \frac{1}{\lambda} + \frac{\lambda_{\mathrm{m}}}{\lambda} ET_{01} + \frac{\lambda_{\mathrm{k}}}{\lambda} ET_{10} = \frac{1}{\lambda}(\lambda_{\mathrm{m}} ET_{01} + \lambda_{\mathrm{k}} ET_{10}).$$

Solving for ET in the above equations yields

$$ET = \frac{1 + \lambda_{\mathrm{m}}/(\lambda + \mu_{\mathrm{m}}) + \lambda_{\mathrm{k}}/(\lambda + \mu_{\mathrm{k}})}{\lambda - \lambda_{\mathrm{m}}\mu_{\mathrm{m}}/(\lambda + \mu_{\mathrm{m}}) - \lambda_{\mathrm{k}}\mu_{\mathrm{k}}/(\lambda + \mu_{\mathrm{k}})} \approx 6.6 \text{ min.}$$