Norwegian University of Science and Technology
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TMA4265 Stochastic processes

Wednesday 2 December 2009 9:00–13:00

Solutions

(Corrected 22 December 2009)

Problem 1

a) The only possibility of getting $X_4 = 2$ given $X_1 = 1$ is $(X_2, X_3, X_4) = (-1, 1, 2)$, which has probability $(1/2)^3 = 1/8$. The only possibilities of getting $X_5 = 2$ given $X_1 = 1$ are the disjoint events $(X_2, X_3, X_4, X_5) = (-1, -2, 1, 2)$ or $(X_2, X_3, X_4, X_5) = (2, -1, 1, 2)$, the union of which has probability $1/16 + 1/16 = 1/8$.

The only possibilities for Anne to have won two rounds in a row in the course of the first five rounds given $X_1 = 1$ are the disjoint events $X_2 = 2$, $(X_2, X_3) = (-1, 1, 2)$ or $(X_2, X_3, X_4, X_5) = (-1, -2, 1, 2)$, the union of which has probability $1/2 + 1/8 + 1/16 = 11/16$.

b) Let m_i denote the expected number of additional rounds needed for one of the players to get three in a row given that one of them has i in a row. Then, by the law of total expectation, $m_1 = \frac{1}{2}(m_2 + 1) + \frac{1}{2}(m_1 + 1)$ and $m_2 = \frac{1}{2} + \frac{1}{2}(m_1 + 1)$, giving $m_1 = 6$. So, after the first round, 6 additional rounds are expected to get three in a row, and the answer is 7.

Let p_i denote the probability that Anne is the first to obtain three wins in a row given that she at the moment has i wins in a row. Then by the law of total probability, $p_1 = \frac{1}{2}p_2 + \frac{1}{2}(1 - p_1)$ and $p_2 = \frac{1}{2} + \frac{1}{2}(1 - p_1)$ (note that $1 - p_1$ is the probability that Anne wins if Bob breaks her row of wins), giving $p_1 = 4/7$.

c) We solve the system $\pi_j = \sum_i \pi_i P_{ij}$ for all j, and $\sum_j \pi_j = 1$. In our case we get $\pi_1 = \alpha(\pi_{-1} + \pi_{-2} + \cdots)$ and $\pi_j = \alpha\pi_{j-1}$ for $j \geq 2$, the latter recursively giving $\pi_j = \alpha^{j-1}\pi_1$. Similarly, $\pi_{-1} = (1-\alpha)(\pi_1 + \pi_2 + \cdots)$ and $\pi_{-j} = (1-\alpha)\pi_{-j+1}$ for $j \geq 2$, the latter recursively giving $\pi_{-j} = (1-\alpha)^{j-1}\pi_{-1}$.

Inserting the π_j in the equation for π_{-1} and summing a geometric series yields $\pi_{-1} = (1-\alpha)(1 + \alpha + \alpha^2 + \cdots)\pi_1 = \pi_1$ (that $\pi_{-1} = \pi_1$ could have been seen more easily by noting that the number of times the process has visited 1 and the number it has visited -1 differ by at most one in any time interval). Requiring the sum of the π_j to be one yields $1 = \pi_1/\alpha + \pi_{-1}/(1-\alpha) = \pi_1(1/\alpha + 1/(1-\alpha))$ (see equations for π_1 and π_{-1} above), or $\pi_1 = \alpha(1-\alpha)$. To summarize, $\pi_j = \alpha^j(1-\alpha)$ and $\pi_{-j} = \alpha(1-\alpha)^j$ for all $j \geq 1$.

The π_j are limiting probabilities. The Markov chain is irreducible. It is also ergodic:

First, the time for return to state 1 if starting in state 1 is the sum of the time it takes from state 1 to state -1, which has the geometric distribution with parameter $1-p$, and the time it takes from state -1 to 1, which has the geometric distribution with parameter p, and the sum of two geometrically distributed variables has finite expectation. Since positive recurrence is a class property, the Markov chain is positive recurrent.

Second, it is possible to get from state 1 to state 1 in any number of transitions greater than or equal to two. So state 1, and thus the entire Markov chain, is aperiodic.

d) Let p and q be the probabilities that Anne will be the first to get n wins in a row given that the Markov chain is in state 1 or -1, respectively. Starting from state 1, the only possibility for Bob to be the first to get n wins in a row is that the Markov chain gets to state -1 before Anne achieves n in a row, and Bob then gets n in a row first, $1 - p = (1-\alpha^{n-1})(1-q)$. A similar argument about Anne, starting in state -1, gives $q = (1-(1-\alpha)^{n-1})p$. $X_1 = 1$ with probability α and $X_1 = -1$ with probability $1-\alpha$. Solving for p and q in the two equations above, we find $\alpha p + (1-\alpha)q = (1-(1-\alpha)^n)\alpha^{n-1}/(1-(1-\alpha^{n-1})(1-(1-\alpha)^{n-1}))$.

Problem 2

a) The instantaneous transition rate from state i to state j is given by $q_{ij} = v_i P_{ij}$, where v_i is the rate of the exponential distribution of state i. This means that $q_{01} = \alpha\lambda$, $q_{02} = (1-\alpha)\lambda$, $q_{10} = \beta\mu_1$, $q_{12} = (1-\beta)\mu_1$, $q_{20} = \mu_2$ and $q_{21} = 0$.

b) Denote by P_0, P_1 and P_2 the limiting probabilities. The balance equations are $\lambda P_0 = \beta\mu_1 P_1 + \mu_2 P_2$, $\mu_1 P_1 = \alpha\lambda P_0$ and $\mu_2 P_2 = (1-\alpha)\lambda P_0 + (1-\beta)\mu_1 P_1$. Solving in terms of P_0 yields $P_1 = \alpha\lambda P_0/\mu_1$ and $P_2 = (\lambda P_0 - \beta\mu_1 P_1)/\mu_2 = \lambda(1-\alpha\beta)P_0/\mu_2$. Since $P_0 + P_1 + P_2 = 1$,

$$1 = P_0\left(1 + \alpha\frac{\lambda}{\mu_1} + \frac{\lambda}{\mu_2}(1-\alpha\beta)\right), \qquad \text{giving} \qquad P_0 = \frac{1}{1 + \frac{\lambda}{\mu_2}\left(1 + \alpha\left(\frac{\mu_2}{\mu_1} - \beta\right)\right)}.$$

With $\lambda = 1/10$, $\mu_1 = 1$ and $\mu_2 = 1/2$ we get $P_0 = 5/6$ for $\beta = 1/2$ and $P_0 = 5/(6 - \frac{\alpha}{10})$ for $\beta = 3/5$.

c) P_0 is maximal when $\alpha(\mu_2/\mu_1 - \beta)$ is minimal. If $\mu_2/\mu_1 < \beta$, we should choose $\alpha = 1$, and if $\mu_2/\mu_1 > \beta$, we should choose $\alpha = 0$ (the choice of α doesn't matter if $\mu_2/\mu_1 = \beta$). So if the ratio between mean quick repair time and mean thorough repair time is greater than the probability of getting the machine back directly from quick repair, then quick repair should be avoided. Otherwise the machine should always be sent to quick repair.

Problem 3

From the cost identity $V = \lambda E(SW_Q^*) + \lambda ES^2/2$, where S is service time, utilizing that S and W_Q^* are independent and that $V = W_Q = EW_Q^*$ for an $M/G/1$ system, we obtain the Pollaczek–Khintchine formula, $W_Q = \lambda ES^2/(2(1-\lambda ES))$. Inserting $ES = \alpha$, $ES^2 = \beta + \alpha^2$ and $\lambda = 2$ we get $W_Q = (\alpha^2 + \beta)/(1 - 2\alpha)$.

By Little's formula, the average number of customers waiting in queue is $L_Q = 2W_Q = 2(\alpha^2+\beta)/(1-2\alpha)$. The mean cost of the system is $2/\alpha + 2(\alpha^2+\beta)/(1-2\alpha)$. The derivative with respect to α is $-2/\alpha^2 + 4(\alpha - \alpha^2 + \beta)/(1-2\alpha)^2$, which is negative whenever $2\alpha^2(\alpha - \alpha^2 + \beta) < (1-2\alpha)^2$. For $\alpha = 1/3$ the derivative is negative when $\frac{2}{9}(\frac{2}{9} + \beta) < \frac{1}{9}$, that is, for $\beta < 5/18$. So increasing α from $1/3$ will reduce mean cost when the variance of the service time is less than $5/18$, $\beta < 5/18$.