Norwegian University of Science and Technology
Department of Mathematical Sciences

English

Contact during examination: Øyvind Bakke
Telephone: 73 59 81 26, 990 41 673

TMA4265 Stochastic processes

Wednesday 2 December 2009 9:00–13:00

Permitted aids: Yellow A5 sheet with your own handwritten notes (stamped by the Department of Mathematical Sciences), *Tabeller og formler i statistikk* (Tapir forlag), *Matematisk formelsamling* (K. Rottmann), calculator HP 30s or Citizen SR-270X

Grades to be announced: 23 December 2009

In the grading each of the eight points counts equally.

In addition to the final examination the project assignment counts 20%.

You should demonstrate how you arrive at your answers (e.g. by including intermediate answers or referral to theory or examples from the reading list).

Pages 2–3 contain problems; pages 4–6 contain formulas.

Problem 1

Anne and Bob play a game that consist of several rounds. Assume that Anne has probability α and Bob has probability $1-\alpha$ of winning a round, $0<\alpha<1$, and that wins are independent of each other. We want to keep track of the number of wins in a row and who has the latest win by modelling the game by means of a discrete-time Markov chain having states ± 1, ± 2, $\ldots$, where $X_n = i > 0$ indicates that Anne has won the last i rounds and $X_n = i < 0$ indicates that Bob has won the last $-i$ rounds, where n is the total number of rounds played.

The transition probabilities are $P_{i,i+1} = \alpha,\ \ P_{i,-1} = 1-\alpha,\ \ P_{-i,-i-1} = 1-\alpha$ and $P_{-i,1} = \alpha$ for all $i \geq 1$, and all other transition probabilities are equal to zero.

In (a) and (b) we assume that $\alpha = \frac{1}{2}$.

a) Find $P(X_4 = 2 \mid X_1 = 1)$ and $P(X_5 = 2 \mid X_1 = 1)$. If Anne wins the first round, what is the probability that she at some time in the course of the first five rounds wins two rounds in a row?

b) Find the expected number of rounds that have been played when the first player obtains three wins in a row. If Anne wins the first round, what is the probability that she is the first to obtain three wins in a row?

In (c) and (d) α may be any number between 0 and 1.

c) Assume that Anne and Bob play the game for many rounds. What are the long-term proportions of time that the Markov chain is in state i, $\ i = \pm 1, \pm 2, \ldots$? Can these proportions be interpreted as limiting probabilities?

d) Find the probability that Anne will be the first player to get n wins in a row.

Problem 2

Each time a machine fails it is sent to quick repair with probability α or to thorough repair with probability $1-\alpha$, $0 \le \alpha \le 1$. Quick repair is faster, but with probability $1-\beta$ it is discovered during quick repair that the machine has to undergo thorough repair.

We model the status of the machine by a three-state continuous-time Markov chain, state 0 meaning that the machine is working, state 1 meaning that it is under quick repair and state 2 that it is under thorough repair. Assume that the amount of time the process spends in a state before making a transition into a different state is exponentially distributed with mean $1/\lambda$ for state 0, $1/\mu_1$ for state 1 and $1/\mu_2$ for state 2, where $\mu_1 > \mu_2$. Transition probabilities P_{ij} from state i to state j are given by $P_{01} = \alpha$, $P_{02} = 1-\alpha$, $P_{10} = \beta$, $P_{12} = 1-\beta$ and $P_{20} = 1$.

a) What are the instantaneous transition rates q_{ij} (from state i to state j)?

b) Show that the long-term proportion of time that the machine is working is

$$P_0 = \frac{1}{1 + \frac{\lambda}{\mu_2}\left(1 + \alpha\left(\frac{\mu_2}{\mu_1} - \beta\right)\right)}.$$

Assume that $\lambda = 1/10$, $\mu_1 = 1$ and $\mu_2 = 1/2$ (per day). Find P_0 in terms of α when (i) $\beta = 1/2$ and (ii) $\beta = 3/5$.

c) State a rule, in terms of general λ, μ_1, μ_2 and β, of how α should be chosen in order to maximize P_0.

Problem 3

Customers arrive at a single server queue according to a Poisson process of rate 2 (per minute) (an $M/G/1$ system). Service times are independent and identically distributed with mean α (minutes) and variance β (min^2). Find W_Q, the mean amount of time a customer spends waiting in queue (in terms of α and β).

The cost of providing the server is $2/\alpha$ kr per minute (whether or not the server is busy), and the cost of keeping a customer waiting in queue is 1 kr per minute (for each customer in the queue). What is the total mean cost per minute for this system? Assume $\alpha = 1/3$. For what values of β will increasing α reduce mean cost?

Formulas

The laws of total probability and total expectation

La B_1, B_2, ... be pairwise disjoint events with $P(\bigcup_{i=1}^{\infty} B_i) = 1$. Then

$$P(A \mid C) = \sum_{i=1}^{\infty} P(A \mid B_i \cap C) P(B_i \mid C), \qquad E(X \mid C) = \sum_{i=1}^{\infty} E(X \mid B_i \cap C) P(B_i \mid C).$$

Discrete-time Markov chains

Chapman–Kolmogorov equations:

$$P_{ij}^{(m+n)} = \sum_{k=0}^{\infty} P_{ik}^{(m)} P_{kj}^{(n)}$$

For an irreducible and ergodic Markov chain the $\pi_j = \lim_{n\to\infty} P_{ij}^n$ exist and are given by the equations

$$\pi_j = \sum_i \pi_i P_{ij} \quad \text{and} \quad \sum_i \pi_i = 1.$$

For transient states i, j and k, the mean time spent in state j given start in state i is

$$s_{ij} = \delta_{ij} + \sum_k P_{ik} s_{kj}.$$

For transient states i and j the probability of ever going into state j given start in state i is

$$f_{ij} = (s_{ij} - \delta_{ij})/s_{jj}.$$

Poisson process

The nth arrival time, S_n, has probability density

$$f_{S_n}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t} \quad \text{for} \quad t \geq 0.$$

Given that the number of events $N(t) = n$, the joint density of S_1, S_2, ..., S_n is

$$f_{S_1,S_2,\ldots,S_n \mid N(t)=n}(s_1, s_2, \ldots, s_n) = \frac{n!}{t^n} \quad \text{for} \quad 0 < s_1 < s_2 < \cdots < s_n \leq t.$$

Continuous-time Markov chains

A Markov process $X(t),\ \ 0 \le t \le \infty$, with state space $\Omega \subseteq \{0, 1, 2, \ldots\}$, is called a birth and death process if

$$\begin{aligned} P(X(t+h) = i+1 \mid X(t) = i) &= \lambda_i h + o(h), \\ P(X(t+h) = i-1 \mid X(t) = i) &= \mu_i h + o(h), \\ P(X(t+h) = i \mid X(t) = i) &= 1 - (\lambda_i + \mu_i)h + o(h), \\ P(X(t+h) = j \mid X(t) = i) &= o(h) \quad \text{when} \quad |j-i| \ge 2, \end{aligned}$$

where $i \ge 0,\ \ \lambda_i \ge 0$ are birth rates and $\mu_i \ge 0$ are death rates.

Chapman–Kolmogorov equations:

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s).$$

Kolmogorov's forward equations:

$$P'_{ij}(t) = \sum_{k \ne j} q_{kj} P_{ik}(t) - v_j P_{ij}(t)$$

Kolmogorov's backward equations:

$$P'_{ij}(t) = \sum_{k \ne i} q_{ik} P_{kj}(t) - v_i P_{ij}(t)$$

If the $P_j = \lim_{t\to\infty} P_{ij}(t)$ exist and are independent of i, the P_j are given by

$$v_j P_j = \sum_{k \ne j} q_{kj} P_k \quad \text{and} \quad \sum_j P_j = 1.$$

For birth and death processes:

$$P_0 = \frac{1}{\sum_{k=0}^{\infty} \theta_k} \quad \text{and} \quad P_k = \theta_k P_0 \quad \text{for } k = 1,\, 2,\, \ldots,$$

where

$$\theta_0 = 1 \quad \text{and} \quad \theta_k = \frac{\lambda_0 \lambda_1 \cdots \lambda_{k-1}}{\mu_1 \mu_2 \cdots \mu_k} \quad \text{for } k = 1,\, 2,\, \ldots$$

Queuing theory

Let L be the average number of customers, L_{Q} average number of customers in queue, W average time a customer spend in the system, W_{Q} average time a customer spend in queue, W_{Q}^* time in queue, S service time, V average remaining work in the system and λ_{a} average arrival rate. Then

$$L = \lambda_{\mathrm{a}} W, \qquad L_{\mathrm{Q}} = \lambda_{\mathrm{a}} W_{\mathrm{Q}}, \qquad V = \lambda_{\mathrm{a}} E(SW_{\mathrm{Q}}^*) + \lambda_{\mathrm{a}} ES^2/2.$$

Some series

$$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}, \qquad \sum_{k=0}^{\infty} k a^k = \frac{a}{(1-a)^2}, \qquad \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n$$

A first order linear differential equations

The differential equation $f'(t) + \alpha f(t) = g(t)$ with the initial condition $f(0) = a$ has solution

$$f(t) = a e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)} g(s)\, ds.$$