Modified 29 September and 6 October 2008

Project – TMA4265 Stochastic Processes Autumn 2008

Professor Ted Hill at the Georgia Institute of Techology used to ask his students to

> either (1) flip a coin 200 times and record the pattern of "heads" or "tails" or (2) simply make up results. The next day he stuns the class by separating the faked data from true trials at a glance. No psychic feat, just a little probability theory at work. A sequence of 200 random coin tosses has a high probability of containing a run of six heads or tails. When people try to fake results, they rarely include such long runs. [1]

We will investigate statistical aspects of getting runs of heads or tails by modelling a sequence of coin flips by a Markov chain.

1. Suggest a suitable finite state Markov chain for investigating occurrences of runs of six or more heads or tails. (Hint: You shouldn't need more than six or seven states.)

Specify the states precisely and the transition probabilities between them. What is the state at the beginning of a sequence of flips?

2. Simulate 10000 sequences of 1000 coin flips, *i.e.* 10000 realizations of the Markov chain.

You may choose whether to save all of the simulations, or to accumulate only quantities that are needed for the following questions. Enclose all computer code used to answer the questions.

In (3–6) below, explain how you used the simulations to obtain estimates, and compare with the analytical results.

3. Find the expected number of flips that is needed to get a run of six heads or tails for the first time. Then use the simulations to estimate this expected number, and also to estimate the standard deviation of the number and to estimate the standard deviation of the estimate of the number.

4. Find the limiting probabilities for being at the end of a run of 1, 2, 3, 4, 5 and ≥ 6 heads or tails analytically. Use the simulations to estimate the probabilities.

[1] `http://gtresearchnews.gatech.edu/reshor/rh-f00/math.html`. See also `http://www.rexswain.com/benford.html` and `http://www.math.ntnu.no/~bakke/TMA4265/2008H/benford.pdf` (in Norwegian).

5. Find the expected number of flips ending a run of 1, 2, 3, 4 and 5 heads or tails before a run of 6 or more occurs for the first time (i.e., find 5 numbers). Compare with your solution of (3) and comment. Verify your calculations by estimating the numbers from the simulations.

6. Find the probability (by calculating an n-step transition probability) that the number of flips it takes to obtain a run of six heads or tails is less than or equal to 200. Verify from the simulations.

7. In general, when making a sequence of coin flips, let T be the number of tosses when a run of k heads or tails occurs for the first time. Show that

$$P(T \leq n) = \begin{cases} 0 & \text{if } n < k, \\ \frac{1}{2^{k-1}} & \text{if } n = k, \text{ and} \\ P(T \leq n-1) + \frac{1}{2^k}(1 - P(T \leq n-k)) & \text{if } n > k. \end{cases}$$

Implement this recursion in computer code to check your answer of (6). Also find $P(T \leq 400)$ for $k = 6$ and for $k = 8$. Make a graph illustrating $P(T \leq n)$ for $k = 6$ and $k = 8$.