



TMA4265 Stochastic processes
ST2101 Stochastic simulation and modelling

Tuesday 9 December 2008 9:00–13:00

Solutions

(Modified 7 January 2009)

Problem 1

- a) When rolling five dice, let the sample space consist of the 6^5 equally probable ordered outcomes. Then P_{05} is the probability that all five dice show the same number after one roll. This event has 6 outcomes, so $P_{05} = 6/6^5 = 1/6^4 = 1/1296$. P_{34} is the probability that exactly one of two dice will show a given number. Then die 1 or die 2 must show the given number, and for the other there are 5 possibilities, so out of the 36 equally probable outcomes for two dice, 10 will give a transition from state 3 to 4, so that $P_{34} = 10/36 = 5/18$.

None of the states communicate, so each equivalence class consists of one state. 0–4 are transient, as there is always a positive probability that we will never return, and 5 is absorbing and thus recurrent.

- b) Getting five of a kind in three rolls is the same as getting to state 5 in 3 steps, and we read $P_{05}^3 = 347897/7558272 \approx 0.046$ from the P^3 matrix.

Getting five of a kind in two additional rolls having saved two alike after the first roll is the same as getting to state 5 in 2 steps starting in state 2, and $P_{25}^2 = 113/3888 \approx 0.029$. So getting only two equal numbers after the first roll is a step backwards in attaining the goal of obtaining five in three rolls.

- c) The expected number of steps to reach state 5 is the sum of the mean time spent in state 1–4 starting from state 0, plus one, that is, the sum of the entries of the first row of $S = (I - P_T)^{-1}$, which is $191283/17248 \approx 11.09$. This could also have been obtained by setting up a system of equations with expected numbers of steps to reach 5 starting in i as unknowns, by conditioning on the initial transition (first step analysis).

If you have saved four equal numbers, then the number of rolls of the last die to get the fifth is geometrically distributed with mean 6. If you have saved three, let m be the expected number to reach five equal numbers. Then conditioning on the first new roll, $m = \frac{25}{36}m + \frac{5}{18} \cdot 6 + 1$ (first step analysis), giving $m = 96/11 \approx 8.73$. Note that these two answers could also have been obtained as the sum of the entries of row 4 and row 5 of S (corresponding to state 3 and 4, respectively), respectively.

- d) The probability that the process ever enters state 1 is $P_{01} = 5/54 \approx 0.093$. The probability that the process ever enters state 3 is $f_{03} = s_{03}/s_{33} = \frac{1500/539}{36/11} = 125/147 \approx 0.85$ (see Formulas and relevant entries of S).

Problem 2

- a) The transition rates are $v_0 = \lambda_0 = 0$ and $v_i = \lambda_i + \mu_i = i(\lambda + \mu)$ for $i \geq 1$. Transition probabilities are $P_{i,i+1} = \lambda_i/(\lambda_i + \mu_i) = i\lambda/(i\lambda + i\mu) = \lambda/(\lambda + \mu)$ and similarly $P_{i,i-1} = \mu/(\lambda + \mu)$. Instantaneous transition rates are given by $q_{ij} = v_i P_{ij}$, that is, $q_{i,i+1} = i\lambda$ and $q_{i,i-1} = i\mu$.

The state 0 is absorbing.

- b) The transition probabilities of Y_n are the P_{ij} of $X(t)$, and let in addition $P_{00} = 1$. Extinction corresponds to $X(t)$ entering state 0 and thus to Y_n entering state 0. The probability of extinction is the probability of Y_n entering 0, that is, the probability that a Gambler's ruin process against an infinitely rich adversary with $p = \lambda/(\lambda + \mu)$ will ever reach 0. Starting in state x_0 , this probability is $((1-p)/p)^{x_0} = (\mu/\lambda)^{x_0}$. (Note that $p > 1/2$ since $\lambda > \mu$.)
- c) This is not a birth and death process, as a transition from any state to 0 is possible. The probability of extinction is 1, as there at any time is the Poisson rate κ of a catastrophe (from state 1 there is the rate ν leading to extinction in addition), so that the time to a catastrophe has the exponential distribution and the catastrophe will eventually take place.

d) Let $Z(t)$ be the population size at time t and let $M(t) = EZ(t)$. Given $Z(t)$,

$$Z(t+h) - Z(t) = \begin{cases} 1 & \text{with probability } Z(t)\lambda h + o(h), \\ -1 & \text{with probability } Z(t)\nu h + o(h), \\ -Z(t) & \text{with probability } \kappa h + o(h), \text{ and} \\ 0 & \text{with probability } 1 - (Z(t)(\lambda + \nu) + \kappa)h + o(h). \end{cases}$$

So

$$E(Z(t+h) - Z(t) \mid Z(t)) = Z(t)(\lambda - \nu)h - Z(t)\kappa h + o(h)$$

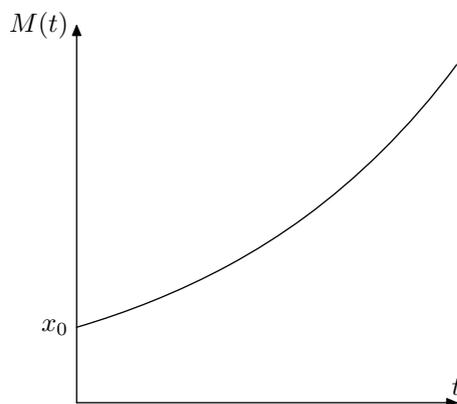
(a small modification of the argument is needed for $Z(t) = 0$ and $Z(t) = 1$, but the latter equation is still valid). Taking expectations yields

$$M(t+h) - M(t) = (\lambda - \nu - \kappa)M(t)h + o(h).$$

Dividing by h and taking the limit as $h \rightarrow 0$ yields the differential equation

$$M'(t) = (\lambda - \nu - \kappa)M(t),$$

which is easily solved, and which is the same equation as the one defining the expected number of animals in the birth and death process of (a) and (b) if $\mu = \nu + \kappa$. So $M(t) = x_0 e^{(\lambda - \nu - \kappa)t}$.



So, if $\mu = \nu + \kappa$, then $EX(t) = EZ(t)$ for all t , yet the $X(t)$ process has a probability of not going extinct, whereas the $Z(t)$ process faces certain extinction.

We get an intuitive explanation of the fact that the two processes have the same expectations by taking a look at the instantaneous transition rates. The $Z(t)$ process has rates $q_{i,i+1} = i\lambda$, $q_{i,i-1} = i\nu$, $q_{i0} = \kappa$, compared with $i\lambda$, $i(\nu + \kappa)$, 0, respectively, for the $X(t)$ process. The processes have the same rates of increase (leading to a higher state), $i\lambda$, but $Z(t)$ has a lower total rate of decrease (leading to a lower state); $i\nu + \kappa$ vs. $i\nu + i\kappa$ for $X(t)$ (and thus also an overall lower state transition rate than $X(t)$ leading to a longer

mean amount of time spent in each state). The effect on the expected number of animals is however balanced by the fact that when the transition from i to 0 occurs, all i animals die, effectively having the same effect on the expectation as an $i\kappa$ death rate in a birth and death process.

We could also argue along the following lines: The $X(t)$ process could have been obtained by each individual giving birth or dying independently with rates λ and μ , respectively. The $Z(t)$ process could have been obtained by each individual giving birth or dying with rates λ and $\nu + \kappa$, respectively. In the latter case deaths are not independent (since a catastrophe will make all individuals die), however, when adding expected values dependence does not matter, so when $\mu = \nu + \kappa$ expected values will be the same.

In conclusion, this example demonstrates that mean values may not be sufficient to describe population behaviour properly.