Norwegian University of Science and Technology
Department of Mathematical Sciences

English

Contact during examination: Øyvind Bakke
Telephone: 73 59 81 26, 990 41 673

TMA4265 Stochastic processes
ST2101 Stochastic simulation and modelling

Tuesday 9 December 2008 9:00–13:00

Permitted aids: Yellow A5 sheet with your own handwritten notes (stamped by the Department of Mathematical Sciences), *Tabeller og formler i statistikk* (Tapir forlag), *Matematisk formelsamling* (K. Rottmann), calculator HP 30s or Citizen SR-270X

Grades to be announced: 9 January 2009

In the grading each of the eight points counts equally.

TMA4265: In addition to the final examination the project assignment counts 20%.
ST2101: In addition to the final examination the project assignment counts 50%.

You should demonstrate how you arrive at your answers (e.g. by including intermediate answers or referral to theory or examples from the reading list).

Pages 2–4 contain problems; pages 5–7 contain formulas.

Problem 1

We play a game with five dice. The goal is to achieve that all five dice show the same number. In the first round we roll five dice. We save dice with the most frequently occurring number. If, for example, the dice show 3, 4, 4, 5, 6, we save the two fours. If there are several numbers occurring with maximal frequency, we choose one number to save – if, for example, the dice show 3, 3, 4, 4, 6, we save either the two threes or the two fours.

In the second round we roll again the dice that we do not save. Again we save the dice (among all five dice) with the most frequently occurring number (we are allowed to change the number we save – if we have saved two twos and the new roll yields three threes, we save the threes).

We continue in this way until all the dice show the same number.

We model the game with a discrete-time Markov chain with states 0, 1, 2, 3, 4, and 5, where $X_0 = 0$ and X_i is the number of dice with the most frequent occurring number after i rolls. If, for example, the dice show 3, 3, 4, 4, 6 after four rolls, then $X_4 = 2$, and if they show 3, 4, 4, 4, 6 after five rolls, then $X_5 = 3$.

The transition matrix of the Markov chain is

$$P = \begin{pmatrix} 0 & \frac{5}{54} & \frac{25}{36} & \frac{125}{648} & \frac{25}{1296} & \frac{1}{1296} \\ 0 & \frac{5}{54} & \frac{25}{36} & \frac{125}{648} & \frac{25}{1296} & \frac{1}{1296} \\ 0 & 0 & \frac{5}{9} & \frac{10}{27} & \frac{5}{72} & \frac{1}{216} \\ 0 & 0 & 0 & \frac{25}{36} & \frac{5}{18} & \frac{1}{36} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is given that

$$P^2 = \begin{pmatrix} 0 & \frac{25}{2916} & \frac{875}{1944} & \frac{28625}{69984} & \frac{8375}{69984} & \frac{221}{17496} \\ 0 & \frac{25}{2916} & \frac{875}{1944} & \frac{28625}{69984} & \frac{8375}{69984} & \frac{221}{17496} \\ 0 & 0 & \frac{25}{81} & \frac{25}{54} & \frac{775}{3888} & \frac{113}{3888} \\ 0 & 0 & 0 & \frac{625}{1296} & \frac{275}{648} & \frac{121}{1296} \\ 0 & 0 & 0 & 0 & \frac{25}{36} & \frac{11}{36} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$P^3 = \begin{pmatrix} 0 & \frac{125}{157464} & \frac{26875}{104976} & \frac{3419375}{7558272} & \frac{115625}{472392} & \frac{347897}{7558272} \\ 0 & \frac{125}{157464} & \frac{26875}{104976} & \frac{3419375}{7558272} & \frac{115625}{472392} & \frac{347897}{7558272} \\ 0 & 0 & \frac{125}{729} & \frac{7625}{17496} & \frac{7375}{23328} & \frac{5359}{69984} \\ 0 & 0 & 0 & \frac{15625}{46656} & \frac{11375}{23328} & \frac{8281}{46656} \\ 0 & 0 & 0 & 0 & \frac{125}{216} & \frac{91}{216} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and that

$$(I - P_{\mathrm{T}})^{-1} = \begin{pmatrix} 1 & \frac{5}{49} & \frac{675}{392} & \frac{1500}{539} & \frac{94575}{17248} \\ 0 & \frac{54}{49} & \frac{675}{392} & \frac{1500}{539} & \frac{94575}{17248} \\ 0 & 0 & \frac{9}{4} & \frac{30}{11} & \frac{965}{176} \\ 0 & 0 & 0 & \frac{36}{11} & \frac{60}{11} \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix},$$

where P_{T} is the matrix obtained by removing the last row and the last column from P, and I is the identity matrix of size 5.

a) Show how the transition probabilities $P_{05} = 1/1296$ and $P_{34} = 5/18$ are calculated.

Which equivalence classes does the chain have? Which equivalence classes are transient and which are recurrent?

b) What is the probability that the dice show five of a kind (five equal numbers) in three or fewer rolls (this corresponds to solving one of the tasks in the dice game *Yahtzee*).

What is the probability that the dice show five of a kind in three or fewer rolls given that you save two alike (two equal numbers) after the first roll?

c) What is the expected number of rolls to get five of a kind?

What is the expected number of additional rolls needed to get five of a kind if you have already saved four of a kind, and what is the expected number of additional rolls needed if you have saved three of a kind?

d) What is the probability that there in the course of the game is a round where all the dice show different numbers? What is the probability that there in the course of the game is a round where exactly three of the dice show the same number?

Problem 2

We assume that an animal population develops as a birth and death process in which birth rates are $\lambda_n = n\lambda$ for $n \geq 0$ and death rates are $\mu_n = n\mu$ for $n \geq 1$. Assume that $\lambda > \mu > 0$. Let $X(t)$ denote the population size at time t.

a) What are the transition rates v_i (from state i to a new state), the transition probabilities P_{ij} and the instantaneous transition rates q_{ij} (from state i to state j)? Does the process have any absorbing states?

We will find the probability that a population starting with x_0 animals at time 0 will go extinct.

Let Y_n be the discrete-time Markov chain that is defined by $Y_0 = X(0)$ and Y_n being the state of $X(t)$ after the nth transition.

b) What are the transition probabilities of the Markov chain Y_n? How can extinction of the population be expressed in terms of this process, and what is the probability that a population starting with x_0 animals will go extinct?

Hint: In a Markov chain having transition probabilities $P_{00} = 1$, $P_{i,i+1} = p = 1 - P_{i,i-1}$ for all $i \geq 1$, that is, *Gambler's ruin* against an infinitely rich adversary, the probability is $((1-p)/p)^i$ that the process ever reaches state 0 given that it starts in state i, if $p > 1/2$.

In a population following this birth and death process and having x_0 animals at time 0, $x_0 e^{(\lambda-\mu)t}$ is the expected number of animals at time t (compare the example of linear growth with immigration, but with the immigration rate set to zero).

We wish to model the possibility of a catastrophe wiping out the entire population, and now assume that the animal population follows a continuous-time Markov chain having instantaneous transition rates $q_{n,n+1} = n\lambda$ for $n \geq 0$, $q_{n,n-1} = n\nu$ and $q_{n0} = \kappa$ for $n \geq 2$, and $q_{10} = \nu + \kappa$, where $\nu > 0$ and $\kappa > 0$. The other transition rates are 0.

c) Is this a birth and death process? What is the probability of extinction in this model?

d) Derive a differential equation for the expected number of animals in the population at time t and solve it. Sketch the graph when $\lambda > \nu + \kappa$.

What is the relation between the expected number of animals in a population following the birth and death process of (a) and (b) if $\nu + \kappa = \mu$? Also compare the probability of extinction, and give a comment on the result.

Formulas

The laws of total probability and total expectation

La B_1, B_2, $\ldots$ be pairwise disjoint events with $P(\bigcup_{i=1}^{\infty} B_i) = 1$. Then

$$P(A \mid C) = \sum_{i=1}^{\infty} P(A \mid B_i \cap C) P(B_i \mid C), \qquad E(X \mid C) = \sum_{i=1}^{\infty} E(X \mid B_i \cap C) P(B_i \mid C).$$

Discrete-time Markov chains

Chapman–Kolmogorov equations:

$$P_{ij}^{(m+n)} = \sum_{k=0}^{\infty} P_{ik}^{(m)} P_{kj}^{(n)}$$

For an irreducible and ergodic Markov chain the $\pi_j = \lim_{n\to\infty} P_{ij}^n$ exist and are given by the equations

$$\pi_j = \sum_i \pi_i P_{ij} \quad \text{and} \quad \sum_i \pi_i = 1.$$

For transient states i, j and k, the mean time spent in state j given start in state i is

$$s_{ij} = \delta_{ij} + \sum_k P_{ik} s_{kj}.$$

For transient states i and j the probability of ever going into state j given start in state i is

$$f_{ij} = (s_{ij} - \delta_{ij})/s_{jj}.$$

Poisson process

The nth arrival time, S_n, has probability density

$$f_{S_n}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t} \quad \text{for} \quad t \geq 0.$$

Given that the number of events $N(t) = n$, the joint density of S_1, S_2, $\ldots$, S_n is

$$f_{S_1,S_2,\ldots,S_n \mid N(t)=n}(s_1, s_2, \ldots, s_n) = \frac{n!}{t^n} \quad \text{for} \quad 0 < s_1 < s_2 < \cdots < s_n \leq t.$$

Continuous-time Markov chains

A Markov process $X(t)$, $0 \le t \le \infty$, with state space $\Omega \subseteq \{0, 1, 2, \ldots\}$, is called a birth and death process if

$$\begin{aligned} P(X(t+h) = i+1 \mid X(t) = i) &= \lambda_i h + o(h), \\ P(X(t+h) = i-1 \mid X(t) = i) &= \mu_i h + o(h), \\ P(X(t+h) = i \mid X(t) = i) &= 1 - (\lambda_i + \mu_i) h + o(h), \\ P(X(t+h) = j \mid X(t) = i) &= o(h) \quad \text{when} \quad |j-i| \ge 2, \end{aligned}$$

where $i \ge 0$, $\lambda_i \ge 0$ are birth rates and $\mu_i \ge 0$ are death rates.

Chapman–Kolmogorov equations:

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s).$$

Kolmogorov's forward equations:

$$P'_{ij}(t) = \sum_{k \ne j} q_{kj} P_{ik}(t) - v_j P_{ij}(t)$$

Kolmogorov's backward equations:

$$P'_{ij}(t) = \sum_{k \ne i} q_{ik} P_{kj}(t) - v_i P_{ij}(t)$$

If the $P_j = \lim_{t \to \infty} P_{ij}(t)$ exist and are independent of i, the P_j are given by

$$v_j P_j = \sum_{k \ne j} q_{kj} P_k \quad \text{and} \quad \sum_j P_j = 1.$$

For birth and death processes:

$$P_0 = \frac{1}{\sum_{k=0}^{\infty} \theta_k} \quad \text{and} \quad P_k = \theta_k P_0 \quad \text{for } k = 1,\, 2,\, \ldots,$$

where

$$\theta_0 = 1 \quad \text{and} \quad \theta_k = \frac{\lambda_0 \lambda_1 \cdots \lambda_{k-1}}{\mu_1 \mu_2 \cdots \mu_k} \quad \text{for } k = 1,\, 2,\, \ldots$$

Queuing theory

Let L be the average number of customers, L_{Q} average number of customers in queue, W average time a customer spend in the system, W_{Q} average time a customer spend in queue, W_{Q}^* time in queue, S service time, V average remaining work in the system and λ_{a} average arrival rate. Then

$$L = \lambda_{\mathrm{a}} W, \qquad L_{\mathrm{Q}} = \lambda_{\mathrm{a}} W_{\mathrm{Q}}, \qquad V = \lambda_{\mathrm{a}} E(S W_{\mathrm{Q}}^*) + \lambda_{\mathrm{a}} E S^2/2.$$

Some series

$$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}, \qquad \sum_{k=0}^{\infty} k a^k = \frac{a}{(1-a)^2}, \qquad \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n$$

A first order linear differential equations

The differential equation $f'(t) + \alpha f(t) = g(t)$ with the initial condition $f(0) = a$ has solution

$$f(t) = a e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)} g(s)\, ds.$$